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The Generalized Linear Complementarity Problem: Least Element Theory and Z-Matrices

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Abstract. Existence of solutions to the Generalized Linear Complementarity Problem (GLCP) is characterized when the associated matrix is a vertical block Z-matrix. It is shown that if solutions exist, then one must be the least element of the feasible region. Moreover, the vertical block Z-matrix belongs to the class of matrices where feasibility implies existence of a solution to the GLCP. The concept of sufficient matrices of class Z is investigated to obtain additional properties of the solution set.

Key words: Complementarity problems, Z-matrices, least element theory

1. Introduction

The linear complementarity problem (LCP) is a well-known NP-complete problem which has polynomially solvable special cases that arise in practice. An extension to this model is the generalized linear complementarity problem (GLCP), which has even more interesting applications than the LCP but is not well understood. In this paper we show that some very important least element and Z-matrix properties of the LCP carry over to the GLCP. In particular, we characterize subclasses of GLCP's that have (unique) solutions and are (polynomially) solvable as linear programs, by showing that some least element results of the LCP carry over to the GLCP.

Let N be an $m \times n$ rectangular block matrix with $m \geq n$. We say that N is a *vertical block matrix* of type $(m_1, \ldots, m_n)$ if N is partitioned row-wise into n blocks such that the jth block of N, N^j, has dimension $m_j \times n$ and $m = \sum_{j=1}^{n} m_j$. The vectors q and w in R^m are also partitioned to conform to the entries in the blocks, N^j, of N:

$$q = \begin{bmatrix} q^1 \\ \vdots \\ q^n \end{bmatrix}, \quad w = \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix},$$

where $q^j\,(q_i^j), w^j = (w_i^j)$ are $m_j \times 1$ matrices.

Given a vertical block matrix N of type $(m_1, \ldots, m_n)$ and $q \in R^m$, the Cottle-Dantzig [3] generalized linear complementarity problem is to find $z \in R^n$ such that

$$Nz + q \geq 0, \quad z \geq 0. \tag{1}$$

$$z_j \prod_{i=1}^{m_j} (N^j z + q^j)_i = 0 \; (j = 1, \ldots, n) \tag{2}$$

Let the above problem be denoted GLCP(q, N).

The fact that the GLCP has tremendous potential to handle complex problems was pointed out by Lemke [13]. In [10] the generalized Leontif input-output linear model is introduced and formulated as a generalized linear complementarity problem. This model is able to address the problem of choosing a new technology and can be further adapted to solve problems related to energy commodity demands, international trade, multinational army personnel assignment, and pollution control. The defining matrix of the model is a vertical block Z-matrix of type $(m_1, \ldots, m_n)$. New programs of research into the GLCP [7–10, 18–20, 23] now begin to show how to realize the potential pointed out by Lemke. Similar to the case of the classical linear complementarity problem, we expect that many of these applications will be defined by a vertical block Z-matrix of type $(m_1, \ldots, m_n)$.

The subject of Complementarity, especially the generalized linear complementarity problem, does not fall into the domain of convex analysis. When such highly nonlinear conditions as in (2) are studied, the techniques of global optimization are generally required. (See [9]). It is important to recognize which cases are also amenable to convex analysis methods. The results of this paper show that linear programming will solve the case of the GLCP associated with the vertical block Z-matrix, and hence, the complexity of more general global optimization techniques will not be a factor in obtaining a solution. Hence the paper obtains the global optimal solution without resorting to complexity beyond the polynomial algorithms of linear programming.

In [3], Cottle and Dantzig showed that if N is a vertical block P- or Strictly Copositive-matrix of type $(m_1, \ldots, m_n)$, then the GLCP(q, N) has a solution. Recent results on characterizations of vertical block Q- and P-matrices and on solution schemes can be found in [8, 9, 19]. Papers by Oh [18] and Isaac and Kostreva [12] develop a related generalized nonlinear complementarity problem.

While complementarity and least element properties of the square Z-matrices have been studied extensively in the literature and algorithms for solving the LCP(q, M) provided, (Chandrasekaran [1], Tamir [21], Mangasarian [15], Cottle, Pang and Venkateswaran [5], and others), not much attention seems to have been paid to the vertical block Z-matrix. This paper provides for the first time the associated properties of vertical block Z-matrices.

The paper is organized as follows. Section 2 is devoted to definitions and notation that will be used throughout the rest of the paper. In Section 3, we give properties and characterizations of the vertical block Z-matrices. Concluding remarks are provided in Section 4.

2. Definitions and Notation

DEFINITION 1. Let M be a real square matrix of size n. M is called a Q-matrix iff LCP(q, M) has a solution for each $q \in R^n$. Similarly, N is a vertical block Q-Matrix of type $(m_1, \ldots, m_n)$ iff the GLCP(q, N) has a solution for each $q \in R^m$

DEFINITION 2. Let N be a vertical block matrix of type $(m_1, \ldots, m_n)$. A submatrix M of N of size n is called a *representative submatrix* if its j-th row is drawn from the j-th block, N^j, of N. A vertical block matrix of type $(m_1, \ldots, m_n)$ has $\prod_{j=1}^n m_j$ representative submatrices. A *principal submatrix* of N is a principal submatrix of some representative submatrix. The determinant of such a matrix is called a principal minor of N.

DEFINITION 3. Let N be a vertical block matrix of type $(m_1, \ldots, m_n)$. N is called a P-matrix iff all its representative submatrices are square P-matrices. The concepts of P_o, Z, M, Copositive, Copositive Plus, and Strictly Copositive vertical block matrices are similarly defined.

3. Matrix Properties

In this section, we generalize some of the well-known properties of the square Z-matrix [21] to the vertical block Z-matrix of type $(m_1, \ldots, m_n)$.

If N is a vertical block Z-matrix, we show in Theorem 1 that feasibility of the GLCP(q, N) implies the existence of complementary solutions.

THEOREM 1. *If N is a vertical block Z-matrix of type* $(m_1, \ldots, m_n)$ *and the set*

$$T_q = \{z : Nz + q \geq 0, z \geq 0\}$$

is nonempty, then the GLCP(q, N) has a solution.

Proof. Let $M_1, \ldots, M_k, \ldots, M_r$ be the representative submatrices of N, where $r = \prod_{j=1}^n m_j$. For each k, let q^k be an $n \times 1$ column vector formed from the components of q corresponding to the rows of N in M_k. Let

$$S^k = \{z^k : M_k z^k + q^k \geq 0, z^k \geq 0\}.$$

Then $T_q \subseteq \cap_{k=1}^r S^k$. Suppose $T_q \neq \emptyset$. Then for each k, $k = 1, \ldots, r$, $S^k \neq \emptyset$. By [21] each LCP(q^k, M_k) has a solution. From [8], there exists a representative submatrix and a vector, say, M_1 and a q^1, such that a solution of the LCP(q^1, M_1) solves the GLC(q, N). This completes the proof. $\square$

In [21] Tamir showed that M is a square Z-matrix if and only if whenever the feasible set for the LCP(q, M) is nonempty, then there exists a solution to the

LCP(q, M) which is also the least element of the set. We extend these results to the GLCP(q, N).

THEOREM 2. *Suppose N is a vertical block matrix of type* $(m_1, \ldots, m_n)$. *Then the following are equivalent.*
(a) N is a vertical block Z-matrix.
(b) For each $q \in R^m$ *for which* T_q *(as defined in Theorem 1) is nonempty, there exists a vector* $\bar{z} \in T_q$, *such that for each* $z \in T_q, \bar{z} \leq z$ *and*

$$\bar{z}_j \prod_{i=1}^{m_j} [(N^j \bar{z})_i + q_i^j] = 0 (j = 1, \ldots, n).$$

Proof. (a)→(b): From Theorem 1, there exists a $(\bar{w}, \bar{z})$ that solves the LCP(q^1, M_1) and the GLCP(q, N), where $\bar{z}$ is the least element of S^1. Since $\bar{z} \in T_q \subset S^1$, then $\bar{z}$ is also the least element of T_q. (b)→(a): If N is not a vertical block Z-matrix of type $(m_1, \ldots, m_n)$, then there exists a block N^j of N such that $N_{ik}^j > 0$, $k \neq j$. Fix i, j and k. Let $q = -N_{0k}$, the k-th column of $-N$. Let e_p denote a $p \times 1$ column vector with each component equal to 1. Let e_p^1 denote the l-th unit vector of dimension p. Then

$$\begin{aligned} Ne_n^k + q &= Ne_n^k - N_{0k}, \\ &= N_{0k} - N_{0k} = 0. \end{aligned}$$

Thus $e_n^k \in T_q$. If $\bar{q} \geq q$, then

$$Ne_n^k + \bar{q} \geq Ne_n^k + q.$$

Thus for all $\bar{q} \geq q$, e_n^k belongs to $T_{\bar{q}} = \{z : Nz + \bar{q} \geq 0, z \geq 0\}$.

Consider the $m \times 1$ column vector defined by

$$\bar{q} = -N_{0k} + \begin{bmatrix} 0 \\ e_{m_k} \\ 0 \end{bmatrix},$$

where m_k is the number of rows of the k-th block. Clearly $\bar{q} \geq q$. Thus $e_n^k \in T_{\bar{q}}$.

Now suppose $T_{\bar{q}} \neq \emptyset$. We claim that under the assumption that N is not a vertical block Z-matrix, the least element of $T_{\bar{q}}$ is not a solution to the GLCP$(\bar{q}, N)$. Assume, for the purpose of contradiction, that it is. So there exists $\hat{z} \in T_{\bar{q}}$ such that for each $z \in T_{\bar{q}}$, $0 \leq \hat{z} \leq z$ and for all $l = 1, \ldots, n$,

$$\hat{z}_l \prod_{i=1}^{m_l} (N^l \hat{z} + \bar{q}^l)_i = 0. \tag{3}$$

In particular, $0 \leq \hat{z} \leq e_n^k$. This implies that for each p, $p \neq k$, $\hat{z}_p = 0$ and $0 \leq \hat{z}_k \leq 1$. For our fixed indices we have that

$$(N^j \hat{z})_i + \bar{q}_i^j = N_{0k}^j \hat{z}_k - N_{0k}^j \geq 0$$

as $\hat{z} \in T_{\bar{q}}$. Since by assumption, $N^j_{ik} > 0$, $j \neq k$, we must have that $\hat{z}_k \geq 1$. Thus $\hat{z}_k = 1$. By the definition of $\bar{q}$,

$$(N^k \hat{z})_i + \bar{q}^k_i = 1 \quad (i = 1, \ldots, m_k).$$

This together with $\hat{z}_k = 1$ contradicts Equation (3). Thus $\hat{z}$ is not a solution to the GLCP$(\bar{q}, N)$. This completes the proof. □

REMARKS: While the formulation in [12] allowed for nonlinear mappings, the linear case in finite dimensions was not considered. Hence no matrix results were obtained at that time.

From the above discussion, a linear programming approach may be used to solve the GLCP(q, N) when N is a vertical block Z-matrix. One may simply use any cost vector with positive components and minimize over the feasible set. This implies that the problem belongs to class P with respect to complexity. Related results appear in [4, 14, 15, 21].

A square matrix M is a Q-matrix if and only if the LCP(q, M) has a solution for each q in R^n. If M is a Z-matrix, this condition reduces to the existence of a unique solution for each q in R^n. In Theorem 4, we give a necessary and sufficient condition under which a vertical block Z-matrix is a Q-matrix. The condition is also equivalent to existence of unique solutions for all q in R^m. It is worthy of note that in applications, for physical considerations, existence of unique solutions is a very desirable property.

THEOREM 3. *Suppose N is a vertical block matrix of type* $(m_1, \ldots, m_n)$. *Then the following are equivalent: (a) N is a vertical block M-Matrix of type* $(m_1, \ldots, m_n)$. *(b) For each q in* R^m*, the polyhedral set* T_q *(as in Theorem 1) is nonempty and there is a unique* $\bar{z}$ *in* T_q *such that* $\bar{z}$ *is the least element of* T_q *and* $\bar{z}_j \prod_{i=1}^{m_j} [(N^j \bar{z})_i + q^j_i] = 0 \quad (j = 1, \ldots, n)$.

Proof. Suppose (a) holds. Then for each q in R^m the GLCP(q, N) has a unique solution [19]. Thus $T_q \neq \emptyset$ for each q. But if for each q in $R^m T_q \neq \emptyset$, then by Theorem 2 there is a $\bar{z}$ in T_q such that $\bar{z}$ is a least element of T_q and

$$\bar{z}_j \prod_{i=1}^{m_j} [(N^j \bar{z})_i + q^j_i] = 0 \quad (j = 1, \ldots, n).$$

by the definition of a vertical block M-matrix. Thus $\bar{z}$ is a unique solution of the GLCP(q, N) which is also the least element of T_q.

Now suppose (b) holds. Then by Theorem 2 N is a vertical block Z-matrix. Moreover, since the GLCP(q, N) has a unique solution for each q in R^m, N is also a P-matrix [19]. Consequently, we have that N is an M-matrix. This completes the proof. □

THEOREM 4. *Let N be a vertical block Z-matrix of type $(m_1, \ldots, m_n)$. Then the following are equivalent:*
(a) N is a vertical block Q-matrix of type $(m_1, \ldots, m_n)$.
(b) N is a vertical block P-matrix $(m_1, \ldots, m_n)$.
(c) The GLCP(q, N) has a solution for some $q < 0$.

Proof. (a)⇒(b): Let M_k be a representative submatrix of N. By [8] M_k is a Q-matrix. Hence the LCP(q^k, M_k) is feasible and solvable for all q^k, in particular, when $q^k < 0$. By [16] M_k is in P.
(b)⇒(c). Obvious.
(c)⇒(a). Let q' be any vector in R^m. Since the GLCP(q, N) has feasible solutions, so does the GLCP(q', N). The result follows from Theorem 1. This completes the proof. □

In a recent paper [5], Cottle, Pang, and Venkateswaran introduced the concept of sufficient matrices for the LCP(q, M). The results for which the associated matrix is Z are extended to the GLCP(q, N). We start with their notation and definitions.

DEFINITION 4. The square matrix M of order n reverses the sign of the vector x in R^n if $x_i(Mx)_i \leq 0$ for all $i = 1, \ldots, n$. Let revM,

$$revM = \{x : x_i(Mx)_i \leq 0, i = 1, \ldots, n\},$$

denote the set of vectors whose sign is reversed by the matrix M.

DEFINITION 5. Let $z, w \in R^n$. The Hadamard product of z and w is defined by

$$(w^t z)_i = w_i z_i, \quad i = 1, 2, \ldots, n,$$

where w^t denotes the transpose of w.

If M is a square matrix of order n, the map $g_M : R^n \rightarrow R^n$ defined by $x \rightarrow x^t Mx$ is the Hadamard product of x and Mx. Observe that $revM = \{x : g_M(x) \leq 0\}$. The kernel of the mapping g_M, denoted $ker g_M$, is the set

$$ker g_M = \{x : g_M(x) = 0\}.$$

We say that a vertical block matrix N has the Hadamard reversal property at x in R^n if for each representative submatrix M, $x_i(Mx)_i \leq 0, i = 1, \ldots, n$.

We let $revN = \cap_{l=1}^{k} \{x : g_{M^l}(x) \leq 0\}$, and $ker\, g_N = \cap_{l=1}^{k} \{x : g_{M^l}(x) = 0\}$, where M^l is the lth representative submatrix of N.

DEFINITION 6. The matrix M in $R^{n \times n}$ is
(i) row sufficient if
$x_i(M^T x)_i \leq 0$ for all i implies $x_i(M^T x)_i = 0$ for all i,
(ii) column sufficient if
$x_j(Mx)_i \leq 0$ for all i implies $x_i(Mx)_i = 0$ for all i,
(iii) sufficient if it is both row and column sufficient.

DEFINITION 7. Let N be a vertical block matrix of type $(m_1, \ldots, m_n)$. Then N is row sufficient if and only if each representative submatrix is row sufficient. The concepts of column sufficient and sufficient vertical block matrices are similarly defined.

Let the feasible set for the GLCP(q, N) be denoted by

$$F(q, N) = \{(w, z) : w = Nz + q, w \geq 0, z \geq 0\}.$$

Define the set $S(q, N)$ by

$$S(q, N) = \left\{(w, z) : (w, z) \in F(q, N), z_j \prod_{i=1}^{m_j} w_i^j = 0, j = 1, \ldots, n\right\}.$$

Then the GLCP(q, N) has a solution iff $S(q, N)$ is nonempty. We assume M is a representative submatrix of N in what follows.

THEOREM 5. *Let N be a vertical block Z-matrix of type $(m_1, \ldots, m_n)$. Then the following are equivalent:*
(i) For all $q \in R^m$, if the GLCP(q, N) is feasible, then $S(q, N)$ is polyhedral.
(ii) For all $J \subseteq \{1, \ldots, m\}, l \subseteq \{1, \ldots, n\}$ for which N_{Jl} is defined as a principal submatrix of N, the system $0 \neq N_{Jl}x_l \leq 0, x_l > 0$ has no solution.
(iii) The matrix N is column sufficient.

Proof. (i)$\rightarrow$(ii): We prove by contradiction. Suppose there exists J, l such that $N_{Jl}x_l \leq 0, x_l > 0$ has a solution $\bar{x}_l > 0$. Define vectors $\bar{q}, \bar{z}$ by the following:

$$\bar{q}_J = -N_{Jl}\bar{z}_l,$$

$$\bar{q}_{J^c} > -N_{J^cl}\bar{z}_l,$$

$$\bar{z}_l = \bar{x}_l, \bar{z}_{l^c} = 0.$$

Then $\bar{q} \geq 0, N\bar{z} + \bar{q} \geq 0$, and

$$\bar{z}_j \prod_{i=1}^{m_j} (N^j\bar{z} + q^j) = 0, \quad j = 1, \ldots, n.$$

Thus $(\bar{w}, \bar{z}) \in S(\bar{q}, N)$. Since $\bar{q} \geq 0, (0, \bar{q}) \in S(\bar{q}, N)$. But this implies that

$$\bar{z}_j \prod_{i=1}^{m_j} (N^j 0 + q^j)_i > 0$$

for j in l, $\bar{z}_l > 0$, $0 \neq \bar{q}_l \geq 0$. Consequently, $S(\bar{q}, N)$ is not polyhedral by Theorem 5 of [5], which also holds for the VLCP. This is a contradiction.
(ii)$\rightarrow$(iii): Let x>0, $x \in revg_N$. If for some i, $x_i(Mx)_i \neq 0$, then $x_i > 0$ and the system in (ii) has a solution which is a contradiction.

(iii)$\rightarrow$(iv): Let $x \in revg_N$. It is sufficient to consider only the case $x_j < 0$, $i = 1, \ldots, n$. Let $y_i = -x_i$. Then $x \leq y$. Since N is a Z-matrix, so is M and $M^j_{ik} \leq 0, k \neq j$. Hence

$$x_i(Mx)_i \geq y_i(My)_i, i = 1, \ldots, m,$$

since $x < 0, y \geq 0$. Thus $y \in revg_N$ and $y \in kerg_N$ by (iii). Hence

$$0 \geq x_i(Mx)_i \geq y_i(My)_i = 0, i = 1, \ldots, m.$$

Therefore, $x \in kerg_N$.
(iv)$\rightarrow$(i): Suppose N is column sufficient. Then each representative submatrix is column sufficient. Let M^i be a representative submatrix of N and q^i a vector appropriately defined from q. If for all q^i in R^n, LCP(q^i, M^i) is feasible, then $S(q^i, M^i)$ is polyhedral by [5]. The implication follows by observing that the above implies for all $q \in R^m$, the GLCP(q, N) is feasible, and that

$$S(q, N) \subseteq S(q^i, M^i)$$

where $i = 1, \ldots, \prod_{j=1}^n m_j$. This completes the proof. $\square$

THEOREM 6. *Let N be a vertical block Z-matrix of type $(m_1, \ldots, n_n)$. Assume that the set $F(q, N)$ is nonempty. Let $\tilde{w} = N\tilde{z} + q$, where $\tilde{z}$ is the least element solving the GLCP(q, N). Then (w, z) belongs to $S(q, N)$ iff $(\tilde{w}, z - \tilde{z})$ belongs to $S(\tilde{w}, N)$.*

Proof. Let (w, z) belong to $S(q, N)$. Then (w, z) is in $F(q, N)$. Thus $\tilde{z} \leq z$. Let $y = z - \tilde{z} \geq 0$. We shall show that $(Ny + q, y)$ belongs to $S(\tilde{w}, N)$. But for any $j = 1, \ldots, n$,

$$N^j y + \tilde{w}^j = N^j y + N^j \tilde{z} + q^j = N^j z + q^j \geq 0 \tag{4}$$

since (w, z) belongs to $S(q, N)$. Consequently,

$$y_j(N^j y + \tilde{w}^j) \geq 0, j = 1, \ldots, n, \tag{5}$$

and there exists an index k such that

$$z_j(N^j z + q^j)_k = 0, 1 \leq k \leq m_j.$$

Now

$$y_j(N^j y + \tilde{w}^j)_k = y_j(N^j y + N^j \tilde{z} + q^j)_k = -\tilde{z}_j(N^j z + q^j)_k \leq 0 \tag{6}$$

since $\tilde{z} \geq 0$, and (w, z) belongs to $S(q, N)$. By (5) and (6), we have that

$$y_j \prod_{i=1}^{m_j} (\tilde{w}^j + N^j y)_i = 0, j = 1, \ldots, n.$$

Therefore, $(Ny+q, y)$ is in $S(\tilde{w}, N)$.

Conversely, let $y = (z - \tilde{z})$ and $(Ny + q, y) \in S(\tilde{w}, N)$. Then $z = y + \tilde{z} \geq 0$. Moreover,

$$Nz + q = Ny + N\tilde{z} + q = Ny + \tilde{w} \geq 0.$$

For each $j, j = 1, \ldots, n$,

$$\begin{aligned} z_j \prod_{i=1}^{m_j} (N^j z + q^j)_i &= (y_j + \tilde{z}_j) \prod_{i=1}^{m_j} (N^j y + N^j \tilde{z} + q^j)_i \\ &= \tilde{z}_j \prod_{i=1}^{m_j} (N^j y + \tilde{w}^j)_i \end{aligned} \tag{7}$$

Since $(\tilde{w}, \tilde{z})$ is the least element of $F(q, N)$, we need to show that if

$$(N^j_{i.} y + \tilde{w}^j_i) > 0, 1 \leq i \leq m_j,$$

then $\tilde{z}_j = 0$. But

$$(N^j_{i.} y + \tilde{w}^j_i) > 0, 1 \leq i \leq m_j,$$

implies that $y_j = 0$ since $(Ny + q, y)$ is in $S(\tilde{w}, N)$. The fact that N is in Z and $y \geq 0$ gives $N_{i.} y \leq 0, 1 \leq i \leq m_j$. Therefore, $\tilde{w}^{j_i} > 0, i = 1, \ldots, m_j$. By the definition of $\tilde{w}$, this implies

$$(N^j_{i.} \tilde{z} + q^j_i) > 0, 1 \leq i \leq m_j.$$

Thus $\tilde{z}_j = 0$. Consequently,

$$\tilde{z}_j \prod_{i=1}^{m_j} (N^j y + \tilde{w}^j)_i = 0 (j = 1, \ldots, n).$$

By (7) and $z \geq 0$, $Nz + q \geq 0$, we conclude that (w, z) is in $S(q, N)$. This completes the proof. □

The properties of the set $S(q, N)$ are interesting from both theoretical and applications point of view. Observe that the results in Theorem 6 formally generalize those of [5] for the LCP and [19] for the GLCP. Thus the class of column sufficient vertical block Z-matrices form an important superset of the set of vertical block M-matrices. It is an open question whether such a GLCP may be completely characterized relative to the geometry of the solution set.

4. Conclusions

Complementarity and least element properties of the vertical block Z-matrices are presented in this paper. It is shown that if N is a vertical block Z-matrix of type $(m_1, \ldots, m_n)$, and the feasible set for the GLCP(q, N) is nonempty, then the problem has a complementary solution. Moreover, if the GLCP(q, N) has more than one solution, then there exists one that is the least element of the feasible region. A necessary and sufficient condition under which the vertical block matrix is a Q-matrix is also provided. The condition is shown to be equivalent to the existence of a unique solution for problem GLCP(q, N) for all q in R^m.

The GLCP(q, N) can be solved using some of the algorithms provided in [3, 8, 9, 15, 19]. However, the least element property of the feasible region implies that the GLCP(q, n) can be solved as a linear program using any strictly positive vector. Thus, if N is in class Z, then the GLCP can be solved in polynomial time.

A strong motivation to study square Z-matrices has been the numerous areas of application. In [10], the authors have exploited the structure of the vertical block Z-matrix to formulate the Generalized Leontif Input-Output Model. The model was successfully applied to the problem of choosing among competing technologies. It is hoped that a better understanding of the properties of the vertical block Z-matrices will enlarge areas of application of the Z-matrices as well as shed new light on existing areas of application.

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